A Spin - 3/2 Ising Model on a Square Lattice

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Abstract

The spin - 3/2 Ising model described by the most general Hamiltonian with an up-down symmetry

$$-\beta H = \sum_{<ij>} \{ J S_i S_j + K S_i^2 S_j^2 + L S_i^3 S_j^3 + \frac{M}{2}(S_i S_j^3 + S_j S_i^3) \} - \Delta \sum_i S_i^2$$

is investigated on a square lattice. It is shown that this model is reducible to an eight - vertex model on a surface in the parameter space spanned by coupling constants J, K, L and M. It is shown that this model is equivalent to an exactly solvable free fermion model along two lines in the parameter space. Consequently, the critical behaviour and, in particular, the critical temperature for the second-order phase transitions of the model was found exactly.

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The Ising model has been one of the most actively studied systems. In particular, spin - 1 and spin - 3/2 Ising models, which present a rich variety of critical and multicritical phenomena, are of special interest.

The spin - 1 Ising model with bilinear (J) and biquadratic (K) nearest- neighbor interactions and a single-ion potential (Δ) is know as the Blume-Emery-Griffiths (BEG) model [1]. The spin - 3/2 Ising model was introduced to explain phase transitions in $DyVO_4$ and its phase diagrams were obtained within the mean-field approximation [2]. Later, this model was used in a study of tricritical properties of a ternary fluid mixture [3]. The complete phase diagram of the spin - 3/2 Ising model model with $L = M = 0$ has been fully analyzed with the use of two different approaches: mean field and Monte-Carlo techniques [4].

Most studies for two-dimensional lattice mentioned above, are performed for the square lattice, very few exact results for spin - 1 and spin - 3/2 Ising models have been carried out only for the trivalent lattices, such as the honeycomb lattice. Recently Horiguchi [5], Wu [6] and Rosengren and Häggkvist [7] using different theoretical approaches have exactly solved the BEG model for the honeycomb lattice in the subspace of interaction constants J and K

$$\exp(K)\cosh J = 1. \tag{1}$$

The same result was obtained also and for the Bethe lattice [8].

Wu and Wu [9] and Kolesik and Samaj [10] have considered the BEG model in an external magnetic field (H) and obtained an exact critical line for all values of H. Recently, Lipowski and Suzuki [11] and Ananikian and Izmailian [12] found the conditions under which the spin - 3/2 Ising model on the honeycomb lattice has the same partition function as the exactly solvable zero-field spin - 1/2 Ising model. Very recently, Horiguchi [13] proposed a general method by which a general spin-S Ising model in expressed in terms of an Ising model of spin ± 1 and spin less then S. This exact results provides an excellent tool for comparing the accuracy of different approximation schemes mentioned above.

In the present paper, we have solved exactly the most general spin - 3/2 Ising model for the two-dimensional square lattice in the subspace of the four- dimentional space spanned by the coupling constants J, K, L and M. We show that this model is reducible to an eight-vertex model, for which the exact solution is not known at the present except for a few special cases [14, 15]. Moreover, we have established the equivalence of our model on the square lattice with one of the special cases, the so-called "free fermion model", which is the eight-vertex model under the "free fermion condition", along two lines in the four-dimentional space spanned by the coupling constants J, K, L and M.

1. We consider the most general spin - 3/2 Ising model with an nearest-neighbor interaction and an up-down symmetry, which is described by the following Hamiltonian

$$-\beta H = \sum_{<ij>} \{ J S_i S_j + K S_i^2 S_j^2 + L S_i^3 S_j^3 + \frac{M}{2}(S_i S_j^3 + S_j S_i^3)\} - \Delta \sum_i S_i^2, \tag{2}$$

where $S_i = \pm\frac{1}{2}, \pm\frac{3}{2}$ is the spin variable at site i and $< ij >$ indicates the summation over the pairs of nearest-neighbor sites.

The equilibrium statistics of the system given by Eq.(2) is determined by the partition function $Z = \sum_{\{s\}} \exp(-\beta H)$, where $\beta = 1/k_B T$ is the inverse temperature and the sum is over all spin configurations.

It is not difficult to show (for details see Ananikian and Izmailian [12]) that if

$$\begin{cases} \tanh^2 J_1 = \tanh J_2 \tanh J_0 \\ \exp(-4K) = \cosh(J_2 - J_0) \end{cases}, \qquad (3)$$

where

$$J_0 = \frac{1}{4}J + \frac{1}{16}M + \frac{1}{64}L, \quad J_1 = \frac{3}{4}J + \frac{15}{16}M + \frac{27}{64}L, \quad J_2 = \frac{9}{4}J + \frac{81}{16}M + \frac{729}{64}L,$$

we can write the following identity

$$\exp\{JS_1S_2 + KS_1^2S_2^2 + LS_1^3S_2^3 + \frac{M}{2}(S_1S_2^3 + S_2S_1^3)\} =$$

$$= \alpha_0 \exp[R(S_1^2 + S_2^2 - \frac{1}{2})]\{1 + 4\tanh J_0 S_1 S_2 \exp[R_0(S_1^2 + S_2^2 - \frac{1}{2})]\}, \qquad (4)$$

where $\alpha_0 = \exp(K/16)\cosh J_0$ and

$$\exp(2R) = \frac{\cosh J_1}{\cosh J_0}\exp(\frac{K}{2}), \quad \exp(4R_0) = \frac{\tanh J_2}{9\tanh J_0}. \qquad (5)$$

Then the partition function defined by Hamiltonian in Eq.(2) can be written as:

$$Z = (\alpha_0 e^{-\frac{R}{2}})^E \sum_{\{s\}} \prod_{<ij>} \{1 + 4\tanh J_0 S_i S_j \exp[R_0(S_i^2 + S_j^2 - \frac{1}{2})]\} \prod_i \exp(-\Delta_0 S_i^2), \qquad (6)$$

where E is the total number of edges, $\Delta_0 = \Delta - \gamma R$, γ is the coordination number of a lattice, and the first product in Eq.(6) is extended over all pairs of neighbouring sites. This result is valid for any arbitrary lattice. It should be noted there that Eq.(3) is an analog of the condition in Eq.(1) for the spin - 1 Ising model.

Thus, we obtain the condition given in Eq.(3), which defines the surface in the space spanned by the coupling constants J, K, L and M, where the partition function is written in the form of Eq.(6). Recently, the equivalence of the spin - 3/2 Ising model on the honeycomb lattice with a zero-field spin - 1/2 Ising model on the same lattice in the subspace given in Eq.(3) have been established [11, 12].

2. In the present section, we investigate the most general spin - 3/2 Ising model on the square lattice, described by the Hamiltonian in Eq.(2). We show that this model is reducible to an eight-vertex model and can be exactly solved on the two nontrivial lines in the surface given by Eq.(3).

Now, consider a square lattice (where the coordination number γ is equal to four) composed of N sites (or vertices) and of $2N$ lattice edges. Then we expand the product $\prod_{<ij>}$ in Eq.(6) and represent graphically each term in the expansion as follows: draw a

dotted or solid line over the lattice edge (ij) if the corresponding term in the expansion contains the factor 1 or $4\tanh J_0 S_i S_j \exp[R_0(S_i^2 + S_j^2 - \frac{1}{2})]$. This leads to eight different kinds of configurations, shown in Fig.1, that can occur at a vertex. We assign a Boltzman weight $\{w_i\}$ to a vertex having $2n$ solid and $4-2n$ broken edges. When, the summations for all sites are carried out, we obtained the following values for weights to vertices:

$$\sum_{s_i=\pm\frac{1}{2},\pm\frac{3}{2}} t^n S_i^n \exp[(nR_0 - \Delta_0)S_i^2] = \begin{cases} a &= 2e^{-\frac{\Delta_0}{4}}[1+e^{-2\Delta_0}], \qquad n=0 \\ b &= 2te^{-\frac{\Delta_0}{4}}[1+9e^{4R_0-2\Delta_0}], \qquad n=2 \\ c &= 2t^2e^{-\frac{\Delta_0}{4}}[1+81e^{8R_0-2\Delta_0}], \quad n=4 \\ &= 0, \qquad n=1,3 \end{cases} \tag{7}$$

where $t = \tanh J_0$ and n is the number of lines with site i as an end-point.

These facts enable us to rewrite Eq.(6) in the form where the sum is over all line configurations on the square lattice having an even number of lines into each site. For the square lattice, this leads to an eight-vertex model shown in Fig.1 with the vertex weights

$$w_1 = a, \quad w_2 = c \quad and \quad w_3 = \ldots = w_8 = b, \tag{8}$$

where a, b and c are given in Eq.(7).

Thus, we obtain the exact equivalence

$$Z = (\alpha_0 e^{-\frac{R_0}{2}})^{2N} Z_{8v}(\{w_i\}), \tag{9}$$

where $Z_8 v(\{w_i\})$ is the partition function of eight-vertex model given by

$$Z_{8v}(\{w_i\}) = 2 \sum_{\text{all line configurations}} \prod_i w_i. \tag{10}$$

The factor of 2 comes from the fact that a reversing of all spins leaves the line configurations unchanged.

The eight-vertex model on the square lattice plays an important role in the study of phase transitions in lattice systems. Unfortunately, except in some special cases [14, 15] the behavior of this general model is not known. The exact expression of the free energy of this model was first obtained by Fan and Wu [15] and by Baxter [14] in respective conditions. In particular, the former authors solved the "free fermion model". This model is defined as a particular case of the eight-vertex model in which the vertex weights satisfy the relation:

$$w_1 w_2 + w_3 w_4 = w_5 w_6 + w_7 w_8, \tag{11}$$

which is called the "free fermion condition", as in the S - matrix formulation of the eight-vertex problem, this condition is equivalent to the consideration of non-interacting many-fermion system [16].

It is readily verified, using Eqs.(7) and (8), that the "free fermion condition" in Eq.(11) is satisfied if, and only if

$$\exp(4R_0) = \frac{1}{9}. \tag{12}$$

This equation together with Eqs.(3) and (5) give us the following two nontrivial lines in the four-dimensional parameter space spanned by coupling constants J, K, L and M, i. e.,

$$\begin{cases} (i) \quad 16J = 49L = -14M, \quad K = 0; \quad (J_0 = \frac{9}{49}J) \\ (ii) \quad 16J = 169L = -26M, \quad K = 0; \quad (J_0 = \frac{36}{169}J) \end{cases}, \tag{13}$$

on which the spin - 3/2 Ising model is equivalent to exactly solvable free fermion model. Here, we remark that except at the trivial point ($J = L = M = K = 0$) for special ($M = 0$) spin - 3/2 Ising model, in general the exactly solvable case can not be obtained [11].

The equivalence, given explicity by Eqs.(9), (10) and (13) permits us to deduce exact analytic properties of the general spin - 3/2 Ising model.

The condition given by Eq.(13) can be also interpretated in another way. Consider, for example, the case (i): Introducing the two new spin variables σ_i and t_i

$$\sigma_i = \frac{4}{3}S_i(S_i^2 - \frac{7}{4}), \quad t_i = S_i^2 - \frac{5}{4} \tag{14}$$

we can write the Hamiltonian given by Eq.(2) in the form

$$-\beta H = \frac{9}{16}L \sum_{<ij>} \sigma_i\sigma_j - \Delta \sum_i t_i - \frac{5}{4}\Delta$$

As it easy to see from Eq.(14) the two states $S_i = \frac{3}{2}, -\frac{1}{2}$ were transformed into $\sigma_i = 1$ and two states $S_i = -\frac{3}{2}, \frac{1}{2}$ into $\sigma_i = -1$, while the states $S_i = \pm\frac{3}{2}$ and $S_i = \pm\frac{1}{2}$ into $t_i = 1$ and $t_i = -1$ respectively. This gives a one-to-one correspondence between S_i and a pair of spins (σ_i, t_i). Sowe can say that the our new spin variables σ_i and t_i are independent and consequently the summation in $\sum_{(\sigma,t)} \exp(-\beta H)$ can be carry out separatly.

Now, we consider the thermodynamic properties of our model. A closed expression for the free energy of the free fermion model is well known [15] and after some algebraic manipulation, we obtain, in the large N limit, the free energy for the spin - 3/2 Ising model on the square lattice in the subspace given by Eq.(13)

$$-\beta f = \ln\{2\sqrt{3}\exp(-\frac{\Delta}{4})[1 + \exp(-2\Delta)]\} + \frac{1}{8\pi^2}\int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \ln[c^2 + s(\cos\vartheta + \cos\varphi)], \tag{15}$$

where $c = \cosh 2J_0$, $s = \sinh 2J_0$ and J_0 is determined from Eq.(13).

Since the second derivative of the free energy in Eq.(15) diverges logarithmically at $s = s_c = 1$, the critical behavior of our model are summarized as follows: The spin - 3/2 Ising model exhibits a first-order phase transition if $s > 1$, a second-order phase transition if $s = 1$ and no transition at all if $s < 1$. Thus, a second-order phase transition occurs at a temperature determined by $\sinh 2J_0 = 1$. This critical condition gives us the two λ - lines in the space spanned by J and Δ

$$(i) \quad J = \frac{49}{18}\ln(1 + \sqrt{2}) = 3,0886... \quad and \quad \Delta - arbitrary,$$

$$(ii) \quad J = \frac{169}{72} \ln(1+\sqrt{2}) = 2,6631... \quad and \quad \Delta - arbitrary,$$

where the spin - 3/2 Ising model exhibits an Ising-type phase transition (logarithmic specific heat singularity).

Besides the intrinsic interest surrounding the spin - S Ising model and possible applications in real physical situations, one is further attracted to the search for obtaining exact nontrivial solutions. So far soluble problems are very few in numbers. In this paper, an exact solution for most general spin - 3/2 Ising model is obtained. It is shown that this model is equivalent to exactly solvable free fermion model along two lines in the parameter space given by Eq.(13). In particular, the analytical expressions for the free energy per spin the λ - lines of Ising-type phase transition were found exactly.

Finally, we point out, that our result suggests that equivalence of spin - 3/2 Ising model to the free fermion model can be extended to a most general Ising model with an half-integer spin. That is, the existence of the exact solvable case for the Ising model with an half-integer spin results from the absence of the $S_i^z = 0$ state.

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